Avalanches of Dry Sand

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Abstract. We present here some general features of sand heaps and of dunes. We mainly discuss avalanche flows, using a modified version of the equations of Bouchaud et al., which might be valid for thick avalanches.

1 Introduction

Granular materials represent a major object of human activities: as measured in tons, the first material manipulated on earth is water ; the second is granular matter (Duran 1996).

This may show up in very different forms : rice, corn, powders for construction (the clinkers which will turn into concrete), pharmaceuticals. . . . In our supposedly modern age, we are extraordinarily clumsy with granular materials. Changing the size, for instance, is difficult: crushing a granular system spends an unreasonable amount of energy, and also leads to an extremely wide distribution of sizes. Transporting a granular material is not easy: sometimes it flows like an honest fluid, but sometimes (in hoppers) it may get stuck: the reopening procedures are complicated -and often dangerous.

Even storage is a problem. The contents of bags can clump. Silos can explode, because of two features :

a) Fine powders of organic materials in air often achieve the optimum ratio of organic/ambient oxygen for detonations.

b) Most grains, when transported, acquire charge by collisions (tribo electricity): high voltages build up, and create sparks.

From a fundamental point of view, granular systems are also very special. The general definition is based on size. We talk of particles which are large enough for thermal agitation to be negligible. Granular matter is a zero temperature system. In practice, Brownian motion may be ignored for particles *larger than on micrometer*: this is our threshold.

A heap of grains is *metastable*: ideally, on a flat horizontal support, it should spread into a monolayer (to decrease its gravitational energy). But it does not! It can be in a variety of frozen states, and the detailed stress distribution, inside the heap, depends on sample history. The dynamics is also very complex: our vision of avalanches is presented in section 2 to 4 -but is probably naive and incomplete.

Not only we do have a great variety of grains: but also a great variety of interactions, commanding the adhesion and the friction between grains. For instance, during dry periods, the grains of sand in a dune have no cohesion, and under the action of wind, the dune moves (Bagnold 1941). In more humid intervals, the grains stick together through minute humidity patches, and they are not entrained by the wind : the dunes stop, thus relieving the plantations from a serious threat. In the present text, we shall concentrate on dry systems, with no cohesion, which give us a relatively well defined model system.

2 Avalanche Problems

2.1 The Coulomb View

C.A. Coulomb (who was at the time a military engineer) noticed that a granular system, with a slope angle θ, larger than a critical value $\theta_{\max}$, would be unstable. He related the angle $\theta_{\max}$ to the friction properties of the material. For granular materials, with negligible adhesive forces, this leads to $tg\theta_{\max} = \mu_i$, where μ_i is a friction coefficient. The instability generates an avalanche. What we need is a detailed scenario for the avalanche.

We note first that the Coulomb argument is not complete: a) it does not tell us at what angle $\theta_{\max} + \epsilon$ the process will actually start b) it does not tell us which gliding plane is preferred among all these of angle $\theta_{\max}$) as shown on Fig. 1.

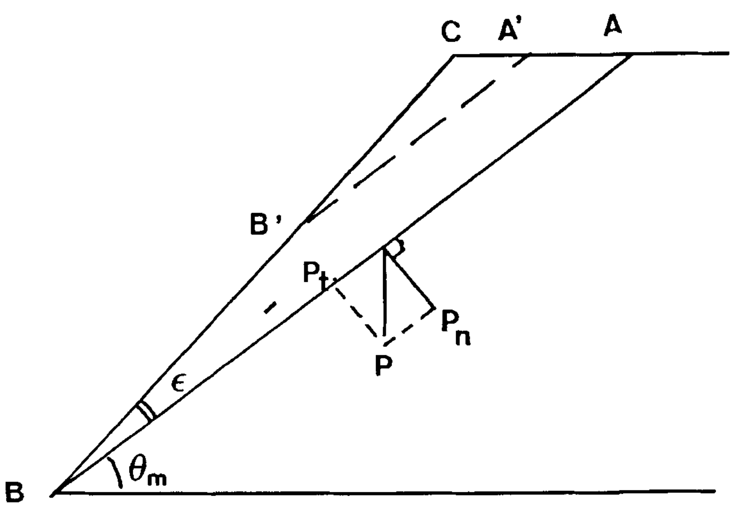

Fig. 1. For a general introduction to granular materials, see J. Duran "Poudres et grains" (Eyrolles, Paris,1996).

I shall propose an answer to these questions based on the notion of a characteristic size ξ in the granular material.

1) Simulations and experiments indicates that the forces are not uniform in a granular medium, but that there are force paths conveying a large fraction of the force. These paths have a certain mesh size ξ, which is dependent on the grain shapes, on the friction forces between them, etc. but which is typically $\xi \sim 5$ to 10 gram diameters d.

2) We also know that, under strong shear, a granular material can display *slip bands*. The detailed geometry of these bands depends on the imposed boundary conditions. But the minimum thickness of a slip band appears to be larger than d. We postulate that the minimum size *coincides with the mesh size* ξ.

We are then able to make a plausible prediction for the onset of the Coulomb process: the thickness of the excess layer must be of order ξ; and the excess angle ϵ must be of order ξ/L, where L is the size of the free surface.

Thus, at the moment of onset, our picture is that a layer of thickness $\sim \xi$ starts to slip. It shall then undergo various processes: (i) the number of grains involved shall be fluidized by the collisions on the underlying heap (ii) it shall be amplified because the rolling grains destabilize some other grains below. The steady state flow has been studied in detailed simulations. It shows a sharp boundary between rolling grains and immobile grains: this observation is the starting point of most current theories.

The amplification process was considered in some detail by Bouchaud et al in a classic paper of 1994 (referred to here at BCRE (Bouchaud et al. 1994; Bouchaud and Cates 1997),(Gennes 1997). It is important to realise that, if we start an avalanche with a thickness ξ of rolling species, we rapidly reach much larger thicknesses R : in practice, with macroscopic samples, we deal with*thick avalanches* ($R >> \xi$). We are mainly interested in these regimes -which, in fact, turn out to be relatively simple.

2.2 Modified BCRE Equations [4]

BCRE discuss surface flow on a slope of profile $h(x,t)$ and slope $tg\theta \simeq \theta = \partial h/\partial x$, with a certain amount $R(x,t)$ of rolling species (Fig. 2). In Ref. (Bouchaud et al. 1994; Bouchaud and Cates 1997), the rate equation for the profile is written in the form:

$$\frac{\partial h}{\partial t} = \gamma R(\theta_n - \theta) \qquad (+\text{ diffusion terms}) \quad . \tag{1}$$

This gives erosion for $\theta > \theta_n$, and accretion for $\theta < \theta_n$.

We call θ_n the neutral angle. This notation differs from BCRE who called itθ_r (the angle of repose). Our point is that different experiments can lead to different angles or repose, not always egal to θ_n.

For the rolling species, BCRE write:

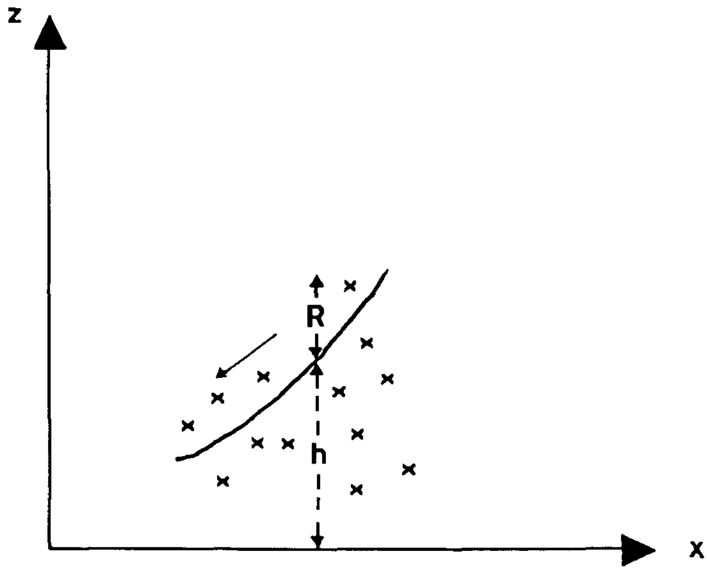

Fig. 2. The basic assumption of the BCRE picture is that there is a sharp distinction between immobile grains with a profile $h(x,t)$ and rolling grains of density $R(x,t)$. R is measured in units of "equivalent height": collision processes conserve the sum $h+R$.

$$\frac{\partial R}{\partial t} = -\frac{\partial h}{\partial t} + v\frac{\partial R}{\partial x} \qquad (+\text{ diffusion terms}) \quad , \tag{2}$$

where γ is a characteristic frequency, and v a flow velocity, assumed to be non vanishing (and approximately constant) for $\theta \sim \theta_n$. For simple grain shapes (spheroidal) and average levels of inelastic collisions, we expect $v \sim \gamma d \sim (gd)^{1/2}$, where d is the grain diameter and g the gravitational acceleration. Eq. (2) gives $\partial h/\partial t$ at as linear in R: this should hold at small R, when the rolling grains act independently. But, when $R > d$, this is not acceptable. Consider for instance the "uphill waves" mentioned by BCRE, where R is constant: Eq. (1) shows that an accident in slope moves upward, with a velocity $v_{\text{up}} = \gamma R$. It is not natural to assume that v_{up} can become very large for large R.

This lead us (namely T. Boutreux, E. Raphaël, and myself) (Boutreux et al. to be published) to propose a modified version of BCRE, valid for flows which involve large R values, and of the form:

$$\frac{\partial h}{\partial t} = v_{\text{up}}(\theta_n - \theta) \qquad (R > \xi) \quad , \tag{3}$$

where v_{up} is a constant, comparable to v. We shall now see the consequences of this modification.

Remark: in the present problems, the diffusion terms in Eq. (2) turn out to be small, when compared to the convective terms (of order d/l, where L is the size of the sample) : we omit them systematically.

2.3 A Simple Case

A simple basic example (Fig. 3) is a two dimensional silo, fed from a point at the top, with a rate $2Q$, and extending over a horizontal span $2L$: the height profile moves upward with a constant velocity Q/L. The profiles were already analysed within the BCRE equations (2,3). With the modified version, the R profile stays the same, vanishing at the wall ($x = 0$) :

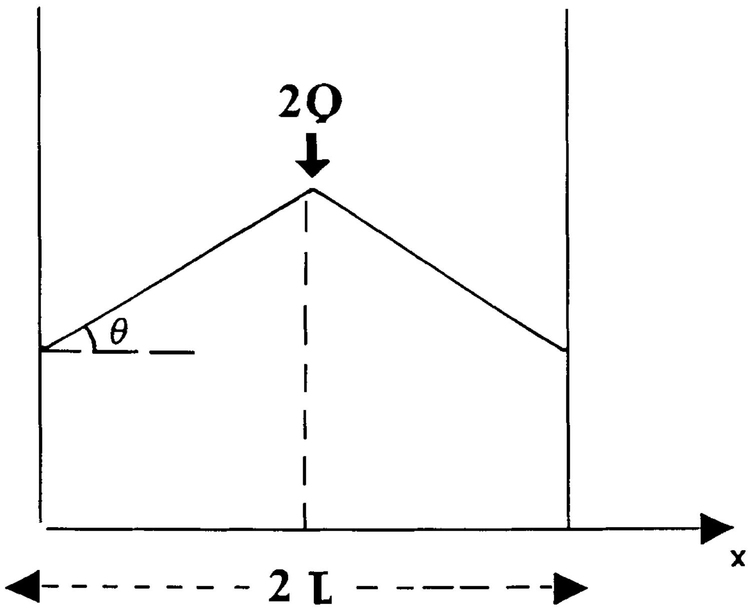

Fig. 3. Feeding of a two dimensional silo with a flux Q over a length L, leading to a growth velocity $w(z) = Q/L$.

$$R = \frac{x}{L}\frac{Q}{v} \; , \tag{4}$$

but the angle is modified and *differs from the neutral angle*: setting $\partial h/\partial t = Q/L$, we arrive at:

$$\theta_n - \theta = \frac{Q}{Lv_{\rm np}} \qquad (Q > v\xi) \; . \tag{5}$$

Thus, we expect a slope which is now dependent on the rate of filling: this might be tested in experiments or in simulations.

3 Downhill and Uphill Motions

Our starting point is a supercritical slope, extending over a horizontal span L with an angle $\theta = \theta_{\max} + \epsilon$ (Fig. 1). Following the ideas of section 1, the excess angle ϵ si taken to be small (of order ξ/L). It will turn out that the exact values of ϵ is not important: as soon as the avalanche starts, the population of rolling species grows rapidly and becomes independent of ϵ (for ϵ small) : this means that our scenarios have a certain level of universality. The crucial feature is that grains roll down, but profiles move uphill : we shall explain this in detail in the next paragraph.

3.1 Wave Equations and Boundary Conditions

It is convenient to introduce a reduced profile

$$\tilde{h}(x,t) = h - \theta_n x \ . \tag{6}$$

Following BCRE, we constantly assume that the angles θ are not very large, and write $tg\theta \sim \theta$: this simplifies the notation. Ultimately, we may write Eqs. (2) and (3) in the following compact form:

$$\frac{\partial R}{\partial t} = v_{\text{up}} \frac{\partial \tilde{h}}{\partial x} + v \frac{\partial R}{\partial x} \ , \tag{7}$$

$$\frac{\partial \tilde{h}}{\partial t} = -v_{\text{up}} \frac{\partial \tilde{h}}{\partial x} \ . \tag{8}$$

Another important condition is that we must have $R > 0$. If we reach $R = 0$ in a certain interval of x, this means that the system is locally frozen, and we must then impose:

$$\frac{\partial \tilde{h}}{\partial t} = 0 \ . \tag{9}$$

One central feature of the modified Eqs. (7, 8) is that, whenever $R > 0$, they are linear. The reduce profile $\tilde{h}$ is decoupled from R, and follows a very simple wave equation:

$$\tilde{h}(x,t) = w(x - v_{\text{up}} t) \ , \tag{10}$$

where w is an arbitrary function describing uphill waves.

It is also possible to find a linear combination of $R(x,t)$ and $\tilde{h}(x,t)$ which moves downhill. Let us put:

$$R(x,t) + \lambda \tilde{h}(x,t) = u(x,t) \ , \tag{11}$$

where λ is an unknown constant. Inserting Eq. (11) into Eq. (2), we arrive at :

$$\frac{\partial u}{\partial t} - v \frac{\partial u}{\partial x} [v_{\text{up}} - \lambda (v_{\text{up}} + v)] \frac{\partial \tilde{h}}{\partial x} \ . \tag{12}$$

Thus, if we choose:

$$\lambda = \frac{v_{\rm up}}{v + v_{\rm up}} \ , \tag{13}$$

we find that u is ruled by a simple wave equation, and we may set

$$u(x,t) = u(x + vt) \ . \tag{14}$$

We can rewrite Eq. (11) in the form:

$$R(x,t) = u(x + vt) - \lambda \ w(x - v_{\rm up}t) \ . \tag{15}$$

Eqs. (10) and (15) represent the normal solution of our problem in all regions where $R > 0$. This formal solution leads in fact to a great variety of avalanche regimes.

3.2 Comparison of Uphill and Downhill Velocities

Our equations introduce two velocities: one downhill (v) and one uphill ($v_{\rm up}$). How are they related ? The answer clearly depends on the precise shape (and surface features) of the grains. Again, if we go to spheroidal grains and average levels of inelasticity, we may try to relate $v_{\rm up}$ and v by a naive scaling argument. Returning to Eq. (1) and (3) for the rate of exchange between fixed and rolling species, we may interpolate between the two limits ($R < \xi$ and $R > \xi$):

$$\frac{\partial h}{\partial t} = \gamma \xi (\theta - \theta_n) f \left(\frac{R}{\xi} \right) \ , \tag{16}$$

where the unknown function f has the limiting behaviours:

$$\left. \begin{array}{l} f(x \to 0) = x \\ \\ f(>> 1) = f_\infty = constant \end{array} \right\} \ . \tag{17}$$

This corresponds to $v_{\rm up} = f_\infty \gamma \xi$. Since we have assumed $v \sim \gamma d$, we are led to:

$$v_{\rm up}/v \sim f_\infty \xi / d \ . \tag{18}$$

If, even more boldly, we assume that $f_\infty \sim 1$, and since ξ is somewhat larger than the grain size, we are led to suspect that $v_{\rm up}$ may be larger than v.

3.3 Closed Versus Open Systems

Various types of boundary conditions can be found for our problems of avalanches :

a) At the top of the heap, we may have a situation of zero feeding ($R = 0$). But we can also have a constant injection rate Q fixing $R = Q/v$. This occurs in the silo of Fig. 3. It also occurs at the top of a dune under a steady wind, where saltation takes place on the windward side (2), imposing a certain injection rate Q, which then induces a steady state flow on the steeper, leeward side.

b) At the bottom end, we sometimes face a solid wall -e.g. in the silo; then we talk about a *closed* cell, and impose $R = 0$ at the wall. But in certain experiments, with a rotating bucket, the bottom end is open (Fig. 4). Here, the natural boundary condition is h = constant at the bottom point, and R is not fixed. Both cases are discussed in Ref. (Boutreux et al. to be published). Here, we shall simply describe some features for the closed cell system.

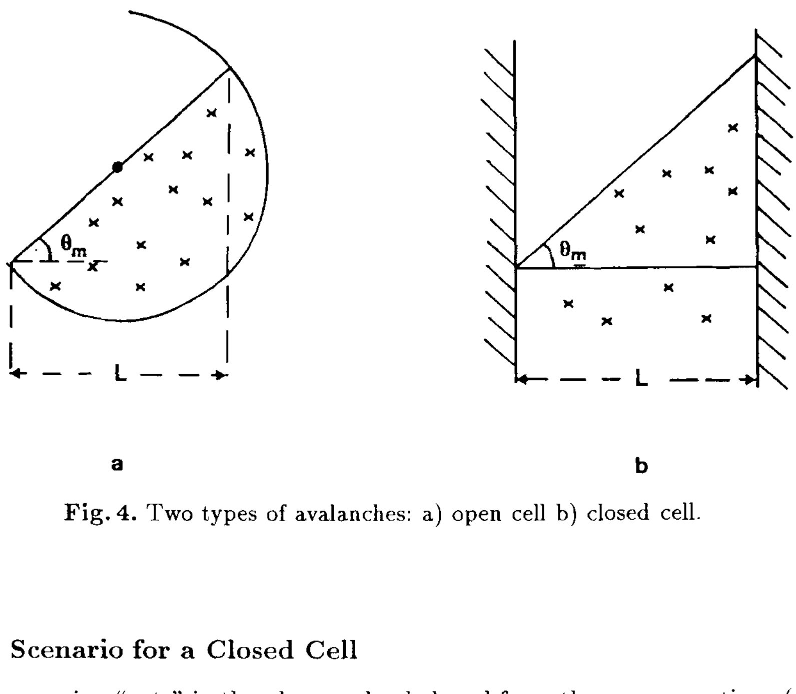

Fig. 4. Two types of avalanches: a) open cell b) closed cell.

3.4 Scenario for a Closed Cell

The successive "acts" in the play can be deduced from the wave equations (10, 15) plus initial conditions. Results are shown in Figs. (5–9). During act I, a rolling wave starts from the top, and an uphill wave starts from the bottom end. In act II, these waves have passed each other. In act III, one of the waves hits the border. If $v_+ > v$, this occurs at the top. From this moment, a region near the top gets frozen, and increases in size. If $v_+ < v$, this occurs at the bottom: the frozen region starts there and expends upwards. In both

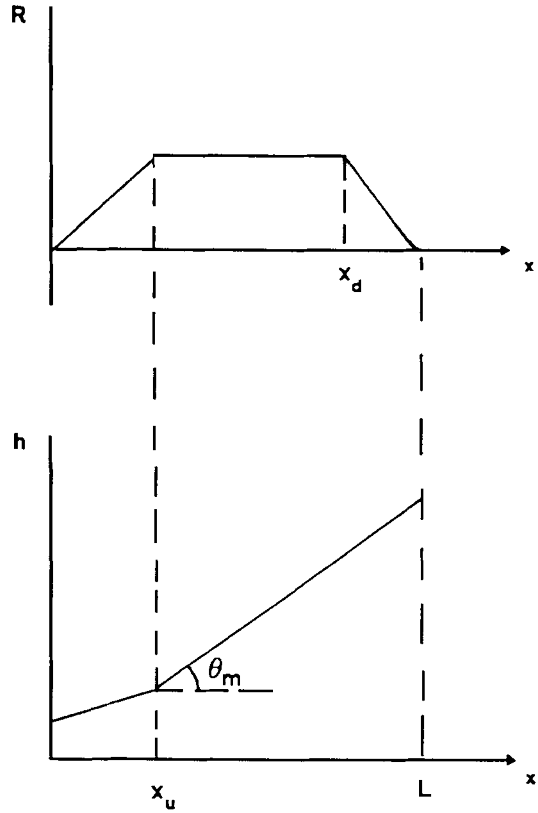

Fig. 5. Closed cell " act I". The slope in the bottom region is described by Eq. (40).

cases, the final slope θ_f is not equal to the neutral angle θ_n, but is *smaller*: $\theta_f = \theta_n - \delta = \theta_{\max} - 2\delta$.

3.5 Discussion

1) The determination of the whole profiles $h(x, t)$ on an avalanche represents a rather complex experiment (Haeger et al. 1988). But certain simple checks could easily be performed.

a) *With an open cell*, the loss of material measured by $R(O, t)$ is easily obtained, for instance, by capacitance measurement (2). The predictions of Ref. (Boutreux et al. to be published) for this loss are described on Fig. 9. $R(O, t)$ rises linearly up to a maximum at $t = L/v$, and then decreases, reaching 0 at the final time $L(1/v + 1/v_{\rm up})$. The integrated amount is:

$$M \equiv \int R(0,t)dt = \frac{1}{2}\lambda\delta L^2 \left(\frac{2}{v} + \frac{1}{v_{\rm up}}\right) . \qquad (19)$$

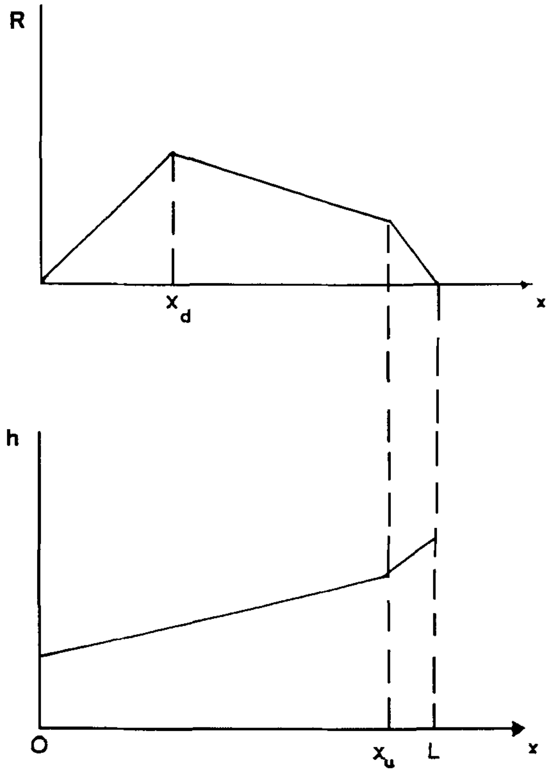

Fig. 6. Closed cell " act II". The sketch has been drawn for $v_{\rm up} > v$. (When $v_{\rm up} < v$, the slope $\partial R/\partial x$, in the central region, becomes positive).

Unfortunately, the attention in Ref. (Haeger et al. 1988) was focused mainly on the reproducibility of M, but (apparently) the value of M and the shape of $R(0,t)$ were not analysed in detail.

b) *With a closed cell*, a simple observable is the rise of the height at the bottom $h(0,t)$: this is predicted to increase linearly with time:

$$h(0,t) = \tilde{h}(0,t) = \delta v_+ t \ , \tag{20}$$

up to $t = L/v_+$, and to remain constant after.

Similar measurements (both for open or closed cells) could be done at the top point, giving $h(L,t)$.

c) A crucial parameter is the final angle θ_f. In our model, this angle is the same all along the slope. For an open cell, it is equal to the neutral angle θ_n. For a closed cell, it is *smaller*: $\theta_f = \theta_n - \delta$.

Thus the notion of an angle of repose is not universal! The result $\theta_f = \theta_n - \delta$ was already predicted in a note (Boutreux and de Gennes), where we

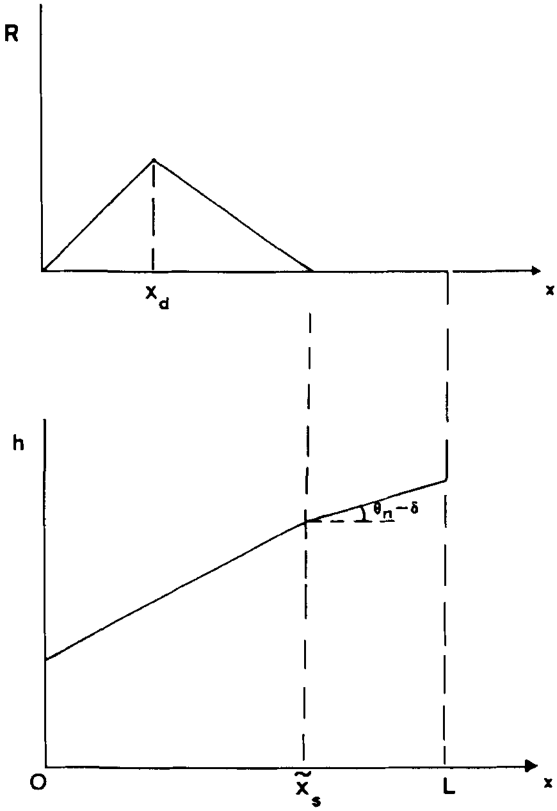

Fig. 7. Closed cell " act III". The case ($v_{up} > v$). A frozen patch grows from the top.

proposed a qualitative discussion of thick avalanches. The dynamics (based on a simplified version of BCRE Eqs.) was unrealistic -too fast- but the conclusion on θ_f was obvious: in a closed cell, the material which starts at the top, has to be stored at the bottom part, and this leads to a decrease in slope.

d) One major unknown of our discussion is the ratio v_{up}/v. We already pointed out that this may differ for different types of grains. Qualitative observations on a closed cell would be very useful here : if in its late stages (act III) the avalanche first freezes at the top, this means $v_{up} > v$. If it freezes from the bottom, v_{up} must be $< v$.

2) Limitations of the present model:

a) Our description is *deterministic*: the avalanche starts automatically at $\theta = \theta_{maxi}$ and sweeps the whole surface. In the open cell systems (with slowly rotating drums) one does find a nearly periodic set of avalanche spikes, suggesting that θ_{max} is well defined. But the amplitude (and the du-

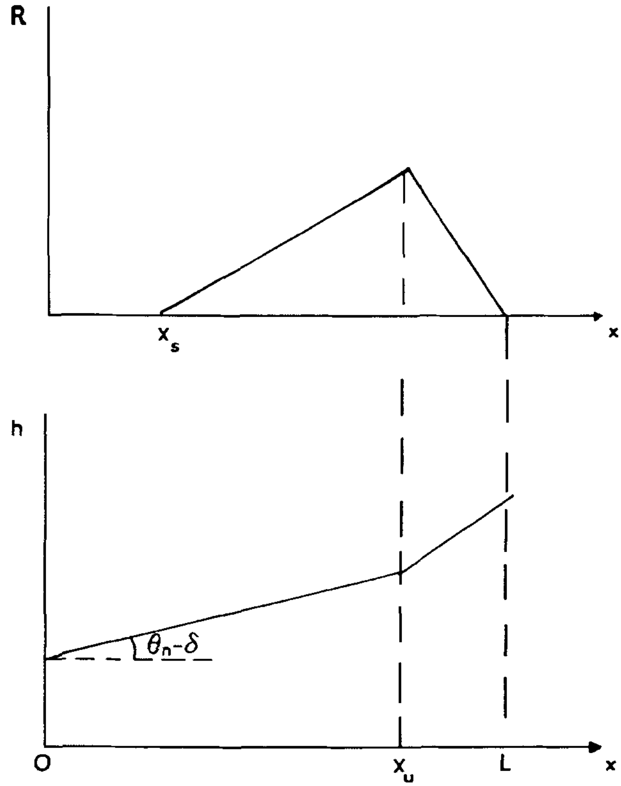

Fig. 8. Closed cell " act III" ($v_{\text{up}} < v$). Here a frozen patch grows from the bottom.

ration) of these spikes varies Boutreux and de Gennes: it may be that some avalanches do not start from the top. We can only pretend to represent the full avalanches.

What is the reason for these statistical features? (i) Disparity in grain size tends to generate spatial in inhomogeneities after a certain number of runs (in the simplest cases, the large grains roll further down and accumulate near the walls). (ii) Cohesive forces may be present: they tend to deform the final profiles, with a $\theta(x)$ which is not constant in space. (iii) Parameters like θ_m (or θ_n) may depend on sample history.

b) *Regions of small* R. For instance, in a closed cell, $R(x,t) \to 0$ for $x \to 0$. A complete solution in the vicinity of $R = 0$ requires more complex equations, interpolating between BCRE and our linear set of equations, as sketched in Eq. (16). Boutreux and Raphaël have indeed investigated this point. It does not seem to alter significantly the macroscopic results described here.

c)*Ambiguities in* θ_n. When comparing thick and thin avalanches, we assumed that θ_n is the same for both: but there may, in fact, be a small dif-

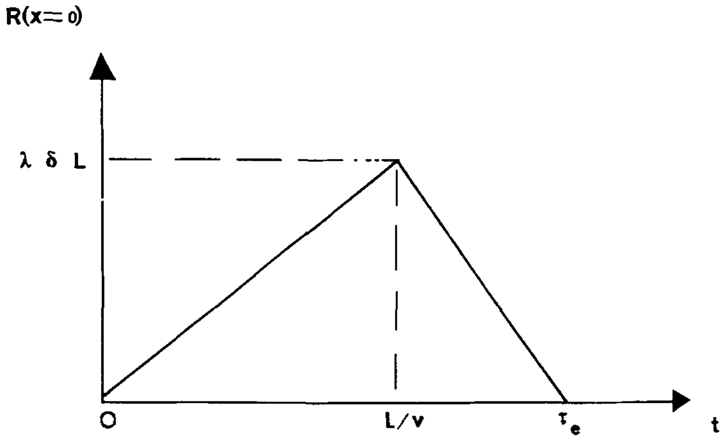

Fig. 9. Flux profile predicted at the bottom of an open cell.

ference between the two. Since most practical situations are related to thick avalanches, we tend to focus our attention on the "thick" case -but this possible distinction between thick and thin should be kept in mind.

Acknowledgments

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