EUROPHYSICS LETTERS 1 October 1999
Europhys. Lett., **48** (1), pp. 49-52 (1999)

Capillary-gravity waves: The effect of viscosity on the wave resistance

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(received 12 March 1999; accepted in final form 8 August 1999)

PACS. 47.35+i – Hydrodynamic waves.
PACS. 68.10−m– Fluid surfaces and fluid-fluid interfaces.

Abstract. – The effect of viscosity on the wave resistance experienced by a two-dimensional perturbation moving at uniform velocity over the free surface of a fluid is investigated. The analysis is based on Rayleigh's linearized theory of capillary-gravity waves. It is shown in particular that the wave resistance remains *bounded* as the velocity of the perturbation approaches the minimum phase speed $c^{\min} = (4g\gamma/\rho)^{1/4}$ (ρ is the liquid density, γ is the liquid-air surface tension, and g the acceleration due to gravity), unlike what is predicted by the inviscid theory.

Consider a body of liquid in equilibrium in a gravitational field and having a planar free surface. If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. This motion will be propagated over the whole surface in the form of waves, which are called *capillary-gravity waves* [1]. These waves are driven by a balance between the liquid inertia and its tendency, under the action of gravity and under surface tension forces, to return to a state of stable equilibrium. For an inviscid liquid of infinite depth, the relation between the circular frequency ω and the wave number k (*i.e.*, the *dispersion relation*) is given by $\omega^2 = gk + \gamma k^3/\rho$, where ρ is the liquid density, γ the liquid-air surface tension, and g the acceleration due to gravity [2]. The above equation may also be written as a dependence of wave speed $c = \omega/k$ on wave number

$$c = (g/k + \gamma k/\rho)^{1/2} . \tag{1}$$

An important feature of eq. (1) is that it implies a minimum phase speed of $c^{\min} = (4g\gamma/\rho)^{1/4}$ reached at $k_{\min} = \kappa$, where $\kappa^{-1} = [\gamma/(\rho g)]^{1/2}$ is the capillary length [3]. For water with $\gamma = 73$ mN m^{-1} and $\rho = 10^3$ kg m^{-3}, the minimun phase speed is $c^{\min} = 0.23$ m s^{-1} and the corresponding wavelength is $\lambda^{\min} = 2\pi/\kappa = 1.7 \times 10^{-2}$ m. The dispersive property of capillary-gravity waves is responsible for the complicated wave pattern generated at the free surface of a still liquid by a disturbance moving with a velocity V greater than $c^{\min}$ [2]. The disturbance may be produced by a small object (such as a fishing line) immersed in the liquid or by the application of an external surface pressure distribution $P_{\rm ext}$. The waves generated by the moving disturbance continuously remove energy to infinity. Consequently, for $V > c^{\rm min}$, the disturbance will experience a drag, R, called the *wave resistance* [4]. For $V < c^{\rm min}$, the wave resistance is equal to zero since, in this case, no waves are generated by the disturbance. A few years ago [5], it has been predicted that the wave resistance corresponding to a surface pressure distribution symmetrical about a point should be *discontinuous* at $V = c^{\rm min}$ [6]. This prediction has been checked very recently by Browaeys and co-workers using a magnetic fluid [7]. The experimental results of Browaeys *et al.* indicate, however, that the disturbance experienced a small but nonzero drag for $V < c^{\rm min}$. Since this nonzero drag might be due, in part, to the finite viscosity of the fluid, it is of some importance to incorporate this physical parameter in the inviscid model of ref. [5]. In order to simplify the discussion, we will here consider the case of a pressure distribution $P_{\rm ext}$ localized along a line. The more complicated case of an axisymmetric surface pressure distribution will be consider elsewhere.

We take the xy-plane as the equilibrium surface of the fluid and assume that a pressure distribution of the form

$$P_{\rm ext} = P_0 \frac{b}{\pi(b^2 + x^2)} \tag{2}$$

travels over the surface in the x-direction with a velocity V (in what follows we assume that $b \ll \kappa^{-1}$). It can be shown that in the case on an inviscid liquid the wave resistance per unit length corresponding to the external pressure distribution (2) is given by [4, 5]

$$R = \frac{P_0^2}{\gamma(k_1 - k_2)} \left[k_1 e^{-2bk_1} + k_2 e^{-2bk_2}\right] \qquad (V > c^{\rm min}) \tag{3}$$

(remember that for an inviscid liquid, $R = 0$ for $V < c^{\rm min}$). In eq. (3), the wave numbers k_1 and k_2 denote the two (real) solutions of $(g/k + \gamma k/\rho)^{1/2} = V$ (see eq. (1)). A brief inspection of eq. (3) shows that the wave resistance is a decreasing function of the perturbation velocity V. In the limit $V \gg c^{\rm min}$, eq. (3) reduces to $R \simeq (P_0^2/\gamma) e^{-4(b/\kappa^{-1})(V/c^{\rm min})^2}$. As the velocity V decreases towards $c^{\rm min}$, the wave resistance eq. (3) becomes unbounded. This result is directly related to the fact that as V approaches $c^{\rm min}$, the energy transferred by the moving pressure distribution cannot be radiated away.

We now turn our attention to the case of a viscous liquid and investigate how the wave resistance (3) is modified by the liquid viscosity. In order to calculate R, we may imagine a rigid cover fitting the surface everywhere, as suggested by Havelock [8]. The assigned pressure system $P_{\rm ext}$ is applied to the liquid surface by means of this cover; hence, the wave resistance is simply the total resolved pressure per unit length in the x-direction [8]:

$$R = -\int P_{\rm ext}(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\zeta(x)\right) \mathrm{d}x, \tag{4}$$

where $\zeta(x)$ denotes (in the frame of the perturbation) the displacement of the free surface from its equilibrium position. Let $\hat{\zeta}$ and $\hat{P}_{\rm ext}$ denote the Fourier transforms of ζ and $P_{\rm ext}$, respectively. Using the Navier-Stokes equation for a viscous fluid along with the appropriate stress condition at the free surface [9], one can relate $\hat{\zeta}$ to $\hat{P}_{\rm ext}$ through the relation

$$\left[(2\nu k^2 - iVk)^2 + g\,|k| + \frac{\gamma}{\rho}|k|^3 - 4\nu^2 |k|^3 \sqrt{k^2 - i\frac{V}{\nu}k}\,\right] \hat{\zeta} = -\frac{|k|}{\rho}\hat{P}_{\rm ext}\,, \tag{5}$$

where the parameter ν is the kinematic viscosity of the liquid. Inserting eq. (5) into eq. (4)

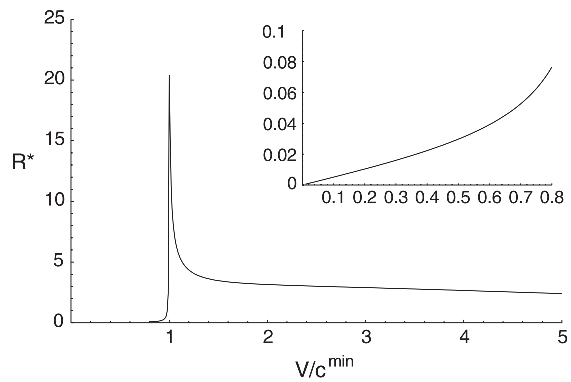

Fig. 1. – $R^* = \pi\gamma\ P_0^{-2} R$ as a function of $v = V/c^{\min}$, with R being the wave resistance —eq. (6), $\epsilon_0 = 3\times10^{-3}$ and $b = 2.5\times10^{-3}\ \kappa^{-1}$. The inset shows the behavior at low velocities.

we obtain

$$R = \frac{P_0^2}{\pi\gamma}\quad \Re\left(\int_0^\infty \frac{ik^2\ \exp\left[-4\frac{b}{\kappa^{-1}}v^2\ k\right]}{(2\epsilon_0\ v\ k^2 - ik)^2 + \frac{1}{4}v^{-4}k + k^3 - 4\epsilon_0^2\ v^2\ k^3\sqrt{k^2 - i\frac{k}{\epsilon_0\ v}}}\,\mathrm{d}k\right), \tag{6}$$

where $v = V/c^{\min}$ and $\epsilon_0 = \eta c^{\min}/\gamma$. The symbol $\Re$ represents the real part of a complex expression. The wave resistance eq. (6) is shown graphically in fig. 1 as a function of the reduced velocity v for $\epsilon_0 = 3\times10^{-3}$ and $b = 2.5\times10^{-3}\ \kappa^{-1}$ (the inset highlights the behaviour of R at low velocities). Two important features should be noted in comparison to the inviscid case: a) first, while R increases steeply near $V = c^{\min}$, it remains bounded; b) secondly, as soon as the perturbation velocity V is greater than zero, the wave resistance takes finite values. This late result is a direct consequence of the internal viscous dissipation inside the liquid.

For ϵ_0 much smaller than unity, eq. (6) can be simplified and the above two features can be recovered analytically (for water with $\gamma = 73$ mM m^{-1} and $\rho = 10^3\,\mathrm{kg\,m^{-3}}$, $\epsilon_0 \approx 3\times10^{-3} \ll 1$). Using standard mathematical techniques [10], one can show that the wave resistance displays a maximum of

$$R = R_{\max}\ \approx \frac{P_0^2}{\gamma\ \sqrt{\epsilon_0}}\,, \tag{7}$$

for

$$V = V_{\max}\ \approx c^{\min}\,(1+\epsilon_0)\,. \tag{8}$$

On the other hand, in the limit $V \ll c^{\min}$, the wave resistance R varies linearly with the perturbation velocity:

$$R \approx \frac{4P_0^2}{\pi\gamma}\,\epsilon_0\left(\frac{V}{c^{\min}}\right)\,\log\left(\frac{\kappa^{-1}}{b}\right)\,. \tag{9}$$

This linear behavior can be observed in the insert in fig. 1.

Let us conclude by a few remarks. In the calculations made above for the wave resistance R, we have used Rayleigh's linearized theory of capillary-gravity waves [3]. We have shown that one of the effects of viscosity was to cutoff the unbounded response of the liquid predicted by the inviscid model as $V \searrow c^{\min}$. For a small viscosity, the response of the liquid remains, however, rather large near $c^{\min}$ (see fig. 1), and it would be of some interest to take nonlinear effects into account in the calculation of R. This is beyond the scope of the present letter. For a recent review of nonlinear capillary-gravity waves, the reader is referred to the work of Dias and Kharif [11]. In the present study we have emphasized the asymptotic behavior of the wave resistance for a liquid of low viscosity. We hope to explore the high-viscosity limit in a subsequent report. Note also that the calculations presented in this letter assumed an external pressure distribution localized along a band. Further work will assess the effect of viscosity for a pressure field symmetrical around a point.

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This work was motivated by discussions with P.-G. de Gennes. We would like to thank him as well as J.-C. Bacri, J. Browaeys, F. Dias and D. Quéré for helpful comments.

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