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Effective Mass of a Charged Particle Travelling above a Dielectric Fluid Surface.

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Abstract. – The behaviour of a charged particle travelling above the surface of a dielectric fluid at constant velocity V is considered. The fluid is taken to be incompressible, inviscid and infinitely deep, and the particle velocity is assumed to be smaller than the minimum phase speed of capillary gravity waves, $c^{\min}$. Because of the electrostatic coupling between the charged particle and the fluid, the particle should induce a bump on the surface. We find that the bump volume Ω is independent of the velocity V of the particle. For $V \ll c^{\min}$, the fluid momentum Q varies linearly with the particle velocity. For larger values of V, however, Q deviates from linearity and diverges like $[1-(V/c^{\min})]^{-1}$ as V approaches $c^{\min}$. From the calculation of Q, we derive the induced mass of the particle, which is *not* directly related to the bump volume.

Introduction. – A great deal of interest in the physical properties of «soft condensed-matter systems» has arisen in recent years [1, 2]. These systems can be defined as molecular systems displaying large responses to small perturbations [3]. «Soft surfaces» might be defined in an analogous way. Recently, one of us proposed a study of the liquid-vacuum interface of low-vapour-pressure dielectric liquids by sending a beam of charged particles above the interface [4]. The method is based on the electrostatic coupling between the charged particles and the dielectric liquid, and consists of measuring the effect of the interface on the particles rather than the surface deformation itself. In this letter we focus on a particular system: a single particle carrying a charge q is placed at a distance d from the surface of an incompressible, inviscid and infinitely deep liquid, and moves at a velocity V parallel to the surface ([1]) [5]. For V smaller than the minimum phase speed of capillary gravity waves $c^{\min} = (4g\gamma/\varrho)^{1/4}$ (where ϱ is the liquid density, γ is the liquid-air surface tension and g is the acceleration due to gravity) [6], the charge should induce a bump on the surface. We find that the bump volume is independent of V. For $V \ll c^{\min}$, the fluid momentum Q varies linearly with the particle velocity. For larger values of V, however, Q deviates from linearity and diverges as V approaches $c^{\min}$. From the calculation of Q, we derive the effective mass of the particle. This study was done within the framework of the linear theory of capillary gravity waves [6].

([1]) This problem was first considered by the authors, see in ref. [5]. However, the calculation of the fluid momentum presented in there was incorrect.

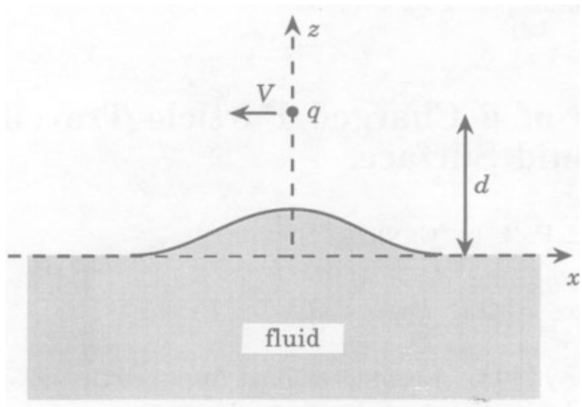

Fig. 1. – Particle of charge q travelling above the surface of a dielectric fluid at constant velocity V. The motion of the particle takes place in the negative x-direction.

Model. – Let us consider the following problem (see fig. 1): a particle of charge q and mass m is travelling at a velocity $\boldsymbol{V}$ above a fluid surface. The fluid is taken to be incompressible, inviscid and infinitely deep. We take the (x, y)-plane as the equilibrium surface of the fluid and the z-axis as the direction perpendicular to the equilibrium surface. The particle velocity may be written in terms of the unit vector $\boldsymbol{u}_x$ in the positive x-direction as $\boldsymbol{V} = -V\boldsymbol{u}_x$, with $V > 0$. The charged particle generates an external surface pressure distribution p_{ext} of the form $p_{\text{ext}}(x, y, t) = P(x + Vt, y)$, with [7]

$$P(x, y) = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2\pi}\frac{\varepsilon_r - 1}{(\varepsilon_r + 1)^2}\frac{\varepsilon_r d^2 + x^2 + y^2}{(d^2 + x^2 + y^2)^3}. \tag{1}$$

Here, ε_r is the relative electric permittivity of the fluid, ε_0 is the electric permittivity of vacuum, and d is the altitude of the particle above the equilibrium surface of the fluid. The total force acting on the fluid surface (the «image force») is directed along the positive z-direction; its magnitude is given by

$$F = -\iint \mathrm{d}x\,\mathrm{d}y\,P(x, y) = \frac{1}{4\pi\varepsilon_0}\frac{\varepsilon_r - 1}{(\varepsilon_r + 1)}\frac{q^2}{4d^2}. \tag{2}$$

Let us assume that the fluid motion is irrotational. Consequently, the fluid velocity can be expressed as $\boldsymbol{v} = \textbf{grad}\,\varphi$, where φ is called the velocity potential. It is determined by solving Laplace's equation, $\Delta\varphi = 0$, along with the boundary conditions

$$\varrho g\frac{\partial\varphi}{\partial z} + \varrho\frac{\partial^2\varphi}{\partial t^2} - \gamma\frac{\partial}{\partial z}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = -\frac{\partial}{\partial t}p_{\text{ext}}, \qquad \text{for } z = 0, \tag{3}$$

and $\partial\varphi/\partial z$ for $z \to -\infty$ [6,8]. Let us seek a velocity potential of the form [6]

$$\varphi(x, y, z, t) = \int\frac{\mathrm{d}k_x}{2\pi}\frac{\mathrm{d}k_y}{2\pi}A(k_x, k_y)\exp\left[i[k_x(x + Vt) + k_y y]\right]\exp[kz], \tag{4}$$

where $k \equiv \sqrt{k_x^2 + k_y^2}$. Equation (4) satisfies both Laplace's equation and the boundary condition for $z \to -\infty$. The amplitude $A(k_x, k_y)$ is obtained by substituting eq. (4) into the free surface boundary condition (3), and is

$$[\gamma k(k^2+\kappa^2)-\varrho V^2 k_x^2]A(k_x, k_y) = -ik_x V\widehat{P}(k_x, k_y), \tag{5}$$

where $\widehat{P}(k_x, k_y)$ denotes the Fourier transform of the function $P(x, y)$ and $\kappa^{-1}=(\gamma/\varrho g)^{1/2}$ is the *capillary length*. Note that since the pressure system $P(x, y)$ (eq. (1)) is symmetrical around the origin, $\widehat{P}(k_x, k_y)$ is a function only of k and can be written as $\widehat{P}(k_x, k_y)=G(k)$, where

$$G(k) = -\frac{q^2}{4\pi\varepsilon_0}\frac{\varepsilon_r-1}{(\varepsilon_r+1)^2}\int_0^{+\infty} \mathrm{d}r\, rJ_0(kr)\frac{\varepsilon_r d^2+r^2}{(d^2+r^2)^3}\,. \tag{6}$$

Here, J_0 denotes the Bessel function of the first kind of zeroth order [9].

Surface displacement. – Let $z=\zeta(x, y, t)$ denote the displacement of the free surface from its equilibrium position. It may be obtained by combining the kinematic relation at the free surface $\partial\zeta/\partial t=(\partial\varphi/\partial z)_{z=0}$ [8] and eq. (4), and is

$$\zeta(x, y, t) = \int\frac{\mathrm{d}k_x}{2\pi}\frac{\mathrm{d}k_y}{2\pi}\widehat{\zeta}(k_x, k_y)\exp\left[i[k_x(x+Vt)+k_y y]\right], \tag{7}$$

where the Fourier component $\widehat{\zeta}(k_x, k_y)$ satisfies

$$\left[\gamma(k^2+\kappa^2)-\varrho V^2\frac{k_x^2}{k}\right]\widehat{\zeta}(k_x, k_y) = -G(k)\,. \tag{8}$$

Since, by assumption, the stream velocity V is smaller than $c^{\min}$, the bracket on the left-hand side of eq. (8) is positive and the bump profile is entirely determined by eq. (8). This regime corresponds to the absence of a wake (see *Conclusion*).

Consider now the volume Ω of the bump (2): $\Omega=\int\int \mathrm{d}x\,\mathrm{d}y\,\zeta(x, y)=\widehat{\zeta}(0, 0)$. Using eq. (8) and the fact that $\kappa^{-2}=(\gamma/\varrho g)$, we obtain

$$\Omega = \frac{F}{\varrho g} = \frac{1}{4\pi\varepsilon_0}\frac{\varepsilon_r-1}{(\varepsilon_r+1)}\frac{q^2}{4\gamma}(\kappa d)^{-2}\,, \tag{9}$$

exactly as if the fluid were at rest. It is remarkable that Ω is independent of the velocity V of the particle. Note that even for d rather large ($d\approx\kappa^{-1}$), Ω can be quite important. For example, for a particle carrying the elementary charge e, the bump volume is of the order of $(e^2/4\pi\varepsilon_0)\gamma^{-1}$—that is, $\Omega\approx 6\cdot 10^3$ Å^3 (3).

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(2) Since the motion of the pressure distribution is steady, we can consider the physical properties of the system at any time including, in particular, $t=0$. In order to simplify the notations, we shall simply write $\zeta(x, y)$ instead of $\zeta(x, y, t=0)$.

(3) We assume the fluid to be characterized by $\gamma\approx 40\cdot 10^{-3}\ \mathrm{N\,m^{-1}}$ and $\varrho\approx 10^3\ \mathrm{kg\,m^{-3}}$; the capillary length is then given by $\kappa^{-1}\approx 2\cdot 10^{-3}$ m.

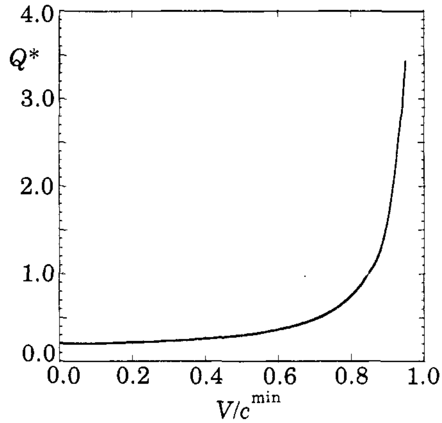

Fig. 2. – $Q^* = (\varrho V \kappa^3 \Omega^2/16)^{-1} Q$ as a function of $V/c^{\min}$ for $\varepsilon_r = 2$ and $d = \kappa^{-1}$, with Q the fluid momentum.

Fluid momentum and effective mass of the particle. – Consider now the momentum of the fluid along the direction of motion (*i.e.* along the negative x-direction):

$$Q = -\int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{\zeta} \mathrm{d}z\, \varrho v_x\,. \tag{10}$$

Since $v_x = \partial\varphi/\partial x$, the x-component of the fluid velocity can be written as (cf. eq. (4))

$$v_x(x, y, z, t) = \int \frac{\mathrm{d}k_x}{2\pi}\frac{\mathrm{d}k_y}{2\pi}\,\widehat{v}_x(k_x, k_y)\exp\left[i[k_x(x+Vt)+k_y y]\right]\exp[kz]\,, \tag{11}$$

where $\widehat{v}_x(k_x, k_y) = -V(k_x^2/k)\,\widehat{\zeta}(k_x, k_y)$. Substituting (11) into eq. (10) and integrating with respect to z, we obtain

$$Q = \varrho V \int \frac{\mathrm{d}k_x}{2\pi}\frac{\mathrm{d}k_y}{2\pi}\frac{k_x^2}{k^2}\,\widehat{\zeta}(k_x, k_y)\int \mathrm{d}x\,\mathrm{d}y\exp\left[i[k_x x + k_y y]\right]\exp[k\zeta(x, y)]\,. \tag{12}$$

We now expand $\exp[k\zeta(x, y)]$ in powers of $k\zeta(x, y)$. It is important to realize that the zeroth-order term does not contribute to Q. This point might be checked by using periodic boundary conditions along the x-coordinate and by noticing that no velocity field is associated with the $k = 0$ mode ($\widehat{v}_x(k = 0) = 0$). Going to the next order, we find

$$Q = \varrho V \int \frac{\mathrm{d}k_x}{2\pi}\frac{\mathrm{d}k_y}{2\pi}\frac{k_x^2}{k}\,|\widehat{\zeta}(k_x, k_y)|^2\,. \tag{13}$$

Inserting eq. (8) into (13), and using eq. (9), we obtain

$$Q = \frac{\varrho V \kappa^3 \Omega^2}{16}\left[\frac{4}{\pi}\int_0^{+\infty} \mathrm{d}u\, \frac{u^2}{(u^2+1)^{1/2}(u^2 - 2(V/c^{\min})^2 u + 1)^{3/2}}\left(\frac{G(u\kappa)}{G(0)}\right)^2\right]. \tag{14}$$

Equation (14) is our central result: it describes the variations of the fluid momentum Q with the particle velocity V. For $V \ll c^{\min}$, the fluid momentum varies linearly with the particle velocity and is given by $\varrho V\kappa^3 \Omega^2$ times a function of κd and ε_r: $Q(V \ll c^{\min}) = \varrho V\kappa^3 \Omega^2 f(\kappa d;\, \varepsilon_r)$ (4). For larger values of V, however, Q deviates from linearity and diverges like $[1 - (V/c^{\min})]^{-1}$ as V approaches $c^{\min}$. This effect is illustrated in fig. 2, where the dependence of $(\varrho V\kappa^3 \Omega^2/16)^{-1} Q$ on the dimensionless velocity $V/c^{\min}$ has been sketched for $\varepsilon_r = 2$ and $d = \kappa^{-1}$.

Suppose now that the particle is accelerated by some external force f (along the negative x-direction) (5). In this process, the liquid momentum will also be increased. Hence, the force f must be equal to the time derivative of the total momentum of the system which is the sum of the momentum mV of the particle and the momentum Q of the liquid: $f = m\,\mathrm{d}V/\mathrm{d}t + \mathrm{d}Q/\mathrm{d}t$. This equation can be rewritten as

$$f = [m + Q'(V)]\,\mathrm{d}V/\mathrm{d}t\,, \tag{15}$$

where $Q'(V)$ denotes the derivative of Q (eq. (14)) with respect to V. The coefficient of $\mathrm{d}V/\mathrm{d}t$ is called the *effective mass* of the particle. It consists of the actual mass of the particle m and the *induced mass*, which, according to eq. (14), is

$$Q'(V) = \frac{\varrho\kappa^3 \Omega^2}{16}[\;]\,, \tag{16a}$$

where

$$[\;] = \frac{4}{\pi}\int_0^{+\infty} \mathrm{d}u \,\frac{u^2(u^2 + 4(V/c^{\min})^2 u + 1)}{(u^2+1)^{1/2}(u^2 - 2(V/c^{\min})^2 u + 1)^{5/2}} \left(\frac{G(u\kappa)}{G(0)}\right)^2. \tag{16b}$$

For $V \ll c^{\min}$, the induced mass is constant: $Q'(V \ll c^{\min}) = \varrho\kappa^3 \Omega^2 f(\kappa d;\, \varepsilon_r)$ (6). Contrary to what was expected in ref. [5], $Q'(V \ll c^{\min})$ is not simply proportional to the bump volume ($Q'(V \ll c^{\min}) \neq \varrho\Omega$). For larger values of V, $Q'(V)$ becomes velocity dependent and diverges like $[1 - (V/c^{\min})]^{-2}$ as V approaches $c^{\min}$. Note that for an electron (charge e, mass m_e), the low velocity limit of the induced mass is of the order of $10^{-11} m_e$ (see footnote (6)).

Conclusion. – In this letter, we considered (within the framework of the linear theory of capillary gravity waves) the behaviour of a charged particle travelling above the surface of a dielectric fluid at constant velocity V. We showed, in particular, that the fluid momentum Q varies linearly with the particle velocity for small V, while it deviates from linearity for larger values of V and diverges as V approaches $c^{\min}$. This divergence is somewhat similar to the behaviour of the momentum of a relativistic particle as its velocity approaches the speed of light [10].

All of our discussion was restricted to velocities V smaller than the minimum phase speed of capillary gravity waves, $c^{\min}$. For a fluid with $\gamma \approx 40\ \mathrm{mN\,m^{-1}}$ and $\varrho \approx 10^3\ \mathrm{kg\,m^{-3}}$, $c^{\min}$ is rather small: $c^{\min} \approx 0.20\ \mathrm{m\,s^{-1}}$. In the opposite case $V > c^{\min}$, a complicated wave pattern is generated at the free surface of the fluid. The waves generated by the moving particle continually remove energy to infinity, and the particle consequently experiences a drag called the *wave resistance* [11]. The analysis of this wave resistance will be published

(4) In the limit of physical interest ($\kappa d \ll 1$), the function f reduces to a constant: $f(\kappa d;\, \varepsilon_r) = 1/16$.

(5) The following analysis is reminiscent of the development given in Landau and Lifshitz concerning the drag force in potential flow past a body, see ref. [8], § 11.

(6) In the limit $\kappa d \ll 1$, $Q'(V \ll c^{\min})$ is given by $Q'(V \ll c^{\min}) = \varrho\kappa^3 \Omega^2/16$.

separately [12]. The discussion was also restricted to an inviscid fluid of infinite depth. While the effect of a finite depth can be incorporated without much difficulty into our model (7), the case of a viscous fluid is more complex and will require a separate study.

On the whole, our general conclusion is that the passing of charged objects above a dielectric fluid might provide a useful tool of investigation by measuring the effect of the interface on the particles rather than the surface deformation itself. The object may be an electron, a charged extremity (analogous to the tip of an atomic-force microscope), or a beam of charged particles.

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(7) In particular, for a fluid of depth h, the bump volume is given by $\Omega = F/\varrho(g - V^2/h)$. Note that Ω is now velocity dependent.

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