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THE COHESIVE ZONE PROBLEM: A COMPARISON BETWEEN DE GENNES' APPROACH AND THE WEIGHT FUNCTION DERIVATION

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Consider the problem of a mode I semi-infinite crack embedded in a linear elastic material with Young's modulus E and Poisson's ratio ν (see e.g.[1]). The crack extends in the negative x-direction with its tip at x = 0, Fig. 1. The applied loading can be simulated by the prescription of the elastic K^A field far away from the tip

$$\sigma^A(x) = \frac{K^A}{\sqrt{2\pi x}}; \quad u^A(x) = \frac{4K^A}{E^*\sqrt{2\pi}}\sqrt{-x} \tag{1}$$

where σ^A and u^A are respectively the tensile stress and the crack displacement in the y-direction along y = 0 (K^A is the applied stress intensity factor). The material parameter E* is given by E* = E for plane stress and E* = E/(1-ν^2) for plane strain. In a cohesive zone model the singular stress is relaxed by inelastic deformation in a zone directly ahead of the crack (0 < x < L) (a thorough investigation of cohesive zone models can be found in [2]). In this region the crack displacement u(x) can be written as

$$u(x) = \frac{4K^A}{E^*\sqrt{2\pi}}\sqrt{L-x} - \frac{2}{\pi E^*}\int_0^L dt\,\sigma(t)\,\mathrm{ln}\left|\frac{\sqrt{(L-x)}+\sqrt{(L-t)}}{\sqrt{(L-x)}-\sqrt{(L-t)}}\right| \tag{2}$$

where $\sigma(x)$ is the actual normal traction in the cohesive zone. This result was obtained by Bueckner [3] and Rice [4] using weight function techniques. It can be shown that the requirement for vanishing square root singularity at x = L is

$$K^A - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^L dt \frac{\sigma(t)}{\sqrt{L-t}} = 0 \tag{3}$$

Recently [5], de Gennes proposed describing the elastic field in the cohesive zone in terms of a source function Φ(x) (with 0 < x < L) defined by

$$\sigma(x) = \frac{1-\nu}{2} E^* \int_0^x dy \Phi(y)(x-y)^{-1/2} \tag{4}$$

$$u(x) = 2(1-\nu) \int_x^L dy \Phi(y)(y-x)^{1/2} \tag{5}$$

In this report we demonstate that this approach is equivalent to the standard weight function formulation (2) and (3).

Our starting point is (4) for the stress field σ(x). Let $\hat{\sigma}(s)$ denote the Laplace transform of σ, i.e.,

$$\hat{\sigma}(s) = L[\sigma(x)]_{(s)} = \int_0^\infty dx \quad e^{-sx}\sigma(x) \tag{6}$$

(4) can be inverted giving

$$\Phi(x) = \frac{2}{\pi(1-\nu)E^*} \int_0^x dy \quad f(y)(x-y)^{-1/2} \tag{7}$$

where

$$f(x) = L^{-1}[s\hat{\sigma}(s)]_{(x)} = \frac{d\sigma(x)}{dx} + \sigma(0^+)\delta(x) \tag{8}$$

Substituting (7) into (5) we obtained

$$\begin{aligned} u(x) &= \frac{4}{\pi E^*} \int_x^L dy (y-x)^{1/2} \int_0^y dt \quad f(t)(y-t)^{-1/2} \\ &= \int_0^x dt \quad J_1(x,t) f(t) + \int_x^L dt \quad J_2(x,t) f(t) \end{aligned} \tag{9}$$

where the functions $J_1(x,t)$ and $J_2(x,t)$ are defined by

$$J_1(x,t) = \int_x^L dy (y-x)^{1/2} (y-t)^{-1/2} \quad (0 < t < x < L) \tag{10}$$

$$J_2(x,t) = \int_t^L dy (y-x)^{1/2} (y-t)^{-1/2} \quad (0 < x < t < L) \tag{11}$$

respectively. It can be shown that within their respective range of definition the two functions $J_1(x,t)$ and $J_2(x,t)$ are both equal to the function K(x,t) defined by

$$K(x,t) = \sqrt{(L-t)(L-x)} - \frac{(x-t)}{2} \ln \left| \frac{\sqrt{(L-x)}+\sqrt{(L-t)}}{\sqrt{(L-x)}-\sqrt{(L-t)}} \right| \tag{12}$$

Hence

$$\begin{aligned} u(x) &= \frac{4}{\pi E^*} \int_0^L dt \quad K(x,t) \quad f(t) \\ &= \frac{4}{\pi E^*} \int_0^L dt \left[K(x,t) \frac{d\sigma(t)}{dt} + \sigma(0+)\delta(t) \right] \\ &= -\frac{4}{\pi E^*} \int_0^L dt \quad \sigma(t) \frac{\partial K(x,t)}{\partial t} \end{aligned} \tag{13}$$

Substituting (12) into (13) we obtain

$$\begin{aligned} u(x) &= \frac{4}{\pi E^*} \int_0^L dt \quad \sigma(t) \left[\sqrt{\frac{(L-x)}{(L-t)}} - \frac{1}{2} \ln \left| \frac{\sqrt{(L-x)}+\sqrt{(L-t)}}{\sqrt{(L-x)}-\sqrt{(L-t)}} \right| \right] \\ &= \frac{4\sqrt{L-x}}{E^* \pi} \int_0^L dt \frac{\sigma(t)}{\sqrt{L-t}} \\ &\quad - \frac{2}{\pi E^*} \int_0^L dt \quad \sigma(t) \ln \left| \frac{\sqrt{(L-x)}+\sqrt{(L-t)}}{\sqrt{(L-x)}-\sqrt{(L-t)}} \right| \end{aligned} \tag{14}$$

Equation (14) is identical to (2) with (3). The equivalence of the de Gennes' approach and the standard weight function formulation is thus proven. Note that (3) is used to establish the equivalence of the two approaches. In general, the weight function approach given by (2) does not require the square root singularity at the cohesive zone tip x = L to be eliminated.

REFERENCES

[1] M. Kanninen and C. Popelar, *Advanced Fracture Mechanics*, Oxford University Press (1985).

[2] L.-O. Fager, J.L. Bassani, C.-Y. Hui and D.-B. Xu, *International Journal of Fracture* 52 (1991) 119-

[3] J.R. Rice, *International Journal of Solids and Structures* 8 (1972) 751.

[4] H.F. Bueckner, *Z. Angew. Math. Mech.* 50 (1970) 529.

[5] P.-G. de Gennes, *J. Phys. Fr.* 50 (1989) 2251; *C.R. Acad. Sci. (Paris)* 309 (1989) 1125; *Canadian Journal of Physics* 68 (1990) 1049.

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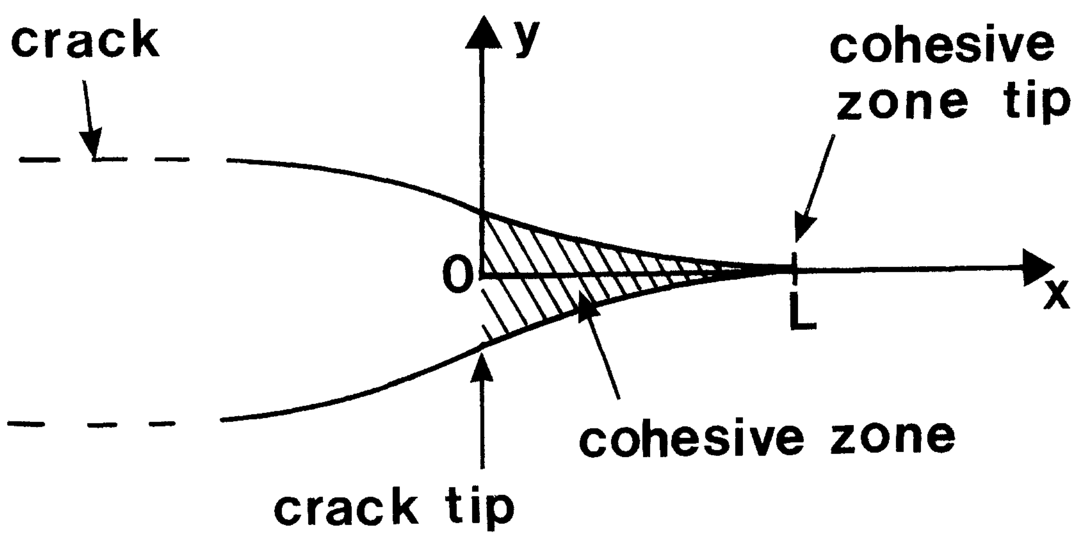

Figure 1. A schematic diagram of a cohesive zone ahead of a crack.